NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

BRITISH MATHEMATICAL OLYMPIAD

Wednesday 17th January 1990

Time allowed – Three and a half hours

PLEASE READ THESE INSTRUCTIONS CAREFULLY

Write on one side of the paper only. ***Use a fresh sheet of paper for each question.*** *Arrange your answers in order. On the first sheet of your script write ONLY your full name, age (in years and completed months on 17th January 1990), home address and school; do not put any working on this sheet.* ***On every sheet of working write your name and initials, your school and the number of the question.***

There is no restriction on the number of questions which may be attempted, but remember

USE FRESH SHEETS FOR EACH QUESTION.

1. Find a positive integer whose first digit is 1 and which has the property that, if this digit is transferred to the end of the number, the number is tripled.

2. $ABCD$ is a square and P is a point on the line AB. Find the maximum and minimum values of the ratio PC/PD, showing that these occur for the points P given by $AP \times BP = AB^2$.

3. The angles A, B, C, D of a convex quadrilateral satisfy the relation

$$\cos A + \cos B + \cos C + \cos D = 0.$$

Prove that $ABCD$ is a trapezium or is cyclic.

4. A coin is biassed so that the probability of obtaining a head is p, $0 < p < 1$. Two players A and B throw the coin in turn until one of the sequences HHH or HTH occurs. If sequence HHH occurs first then A wins. If HTH occurs first then B wins. For what value of p is this game fair (ie. such that A and B have an equal chance of winning)?

[Turn over

5. The diagonals of a convex quadrilateral $ABCD$ intersect at O. The centroids of triangles AOD and BOC are P and Q; the orthocentres of triangles AOB and COD are R and S. Prove that PQ is perpendicular to RS.
[The *centroid* of a triangle is the intersection of the lines joining each vertex to the midpoint of the opposite side; the *orthocentre* is the intersection of the altitudes].

6. Prove that if x, y are rational numbers satisfying the equation
$$x^5 + y^5 = 2x^2y^2$$
then $1 - xy$ is the square of a rational number.

REMEMBER: A FRESH SHEET FOR EACH QUESTION
WITH NAME, SCHOOL AND QUESTION
NUMBER ON EVERY SHEET.

As a result of their performance in BMO, some candidates will be selected to take a **Further International Selection Test** (FIST) early in March 1990. Following this, about 20 candidates will be selected to attend a residential **Training Session** to be held at Trinity College, Cambridge, from 5th - 8th April 1990. The UK team for the 1990 **International Mathematical Olympiad** (IMO) in Beijing will then be chosen.

Prizes: There will be a number of prizes, in the form of book tokens, for those who do very well in BMO. In addition, Trinity College, Cambridge, will award £50 prizes to all those who attend the residential Training Session.

The 31st International Mathematical Olympiad will be held in Beijing, People's Republic of China, beginning 11th July, 1990.

SPONSORS of the British team for the International Mathematical Olympiad:

The Department of Education and Science; Trinity College, Cambridge; The Corporation of the City of London; ICI; The Royal Society; Eton College; Portsmouth Grammar School; St. Paul's School; Westminster School; St. John's College, Cambridge; Dulwich College; The London Mathematical Society; The Manchester Grammar School; Winchester College; Merton College, Oxford.